International Journal of Theoretical Physics, Vol. 35, No. 1, 1996

Centrosymmetric Lie Algebras and Boost-Dilations

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Received April 4, 1995

We define the Lie algebra $c(n)$ of centrosymmetric matrices. It generates a noncompact and nonsemisimple local Lie group with the unusual property that $\exp c(n) \subset c(n)$. The group contains an invariant subgroup of Lorentz boost/dilation transformations. For n even, these form a subgroup of the conformal group of the Lorentzian metric with signature $(- + - + \cdots - +)$.

1. INTRODUCTION

In the general linear (real) Lie algebra $gl(n)$ of all $n \times n$ $(n \geq 2)$ matrices, we will define a subalgebra $c(n)$ of centrosymmetric matrices. The starting point is the matrix

$$J_{ab} = \delta_{a,n+1-b} \tag{1}$$

whose elements vanish except for 1's along the reverse (or minor) diagonal. J is idempotent, symmetric (and consequently orthogonal), and has determinant ± 1:

$$J^2 = I, \qquad J^T = J, \qquad \det(J) = (-1)^{n(n-1)/2} \tag{2}$$

It follows from (2a) that

$$e^{\alpha J} = I \cosh \alpha + J \sinh \alpha \tag{3}$$

so that in particular, J generates the Lorentz Lie algebra when $n = 2$:

$$\text{span}\{J\} = so(1, 1) \qquad (n = 2) \tag{4}$$

Now we use J to define the mapping

$$X \to X^C = JXJ \Rightarrow X^C_{ab} = X_{n+1-a,n+1-b} \tag{5}$$

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0020-7748/96/0100-0129$09.50/0

where the implication follows from (1). Centrosymmetric matrices (Weaver, 1985) are those invariant under (5) (thus showing a reflection symmetry about the center of the matrix):

$$c(n) = \{X \in gl(n) | X^C = X\} \tag{6}$$

J itself is obviously an element of $c(n)$. Skew centrosymmetric matrices form the set

$$\bar{c}(n) = \{X \in gl(n) | X^C = -X\} \tag{7}$$

It is clear that (5) is a linear and algebraic isomorphism of $gl(n)$:

$$(\alpha X + \beta Y)^C = \alpha X^C + \beta Y^C, \qquad (XY)^C = X^C Y^C \Rightarrow [X, Y]^C = [X^C, Y^C] \tag{8}$$

where α, β are scalars. It follows from (6)–(8) that $c(n)$ *is a matrix Lie algebra* (equivalently, a faithful representation of an underlying abstract Lie algebra), while $\bar{c}(n)$ is an associative algebra under matrix product that is not closed under the Lie product (commutator). However, the commutator of skew centrosymmetric matrices is centrosymmetric, so that the derived algebra $\bar{c}(n)'$ is a Lie subalgebra of $c(n)$:

$$\bar{c}(n)' \equiv [\bar{c}(n), \bar{c}(n)] < c(n) \tag{9}$$

To our knowledge, $c(n)$ has not been previously defined or investigated. In Section 2 we show that $c(n)$ is noncompact and nonsemisimple, that it contains its exponential, and that $c(2m)$ contains a boost-dilation subgroup of the conformal group of the Lorentzian metric with signature zero. In Section 3 we consider in detail $c(2)$ and the nonsolvable algebra $c(3)$, whose derived algebra is isomorphic to $sl(2, R)$. We extend the proof of nonsolvability to $c(2m + 1)$, $m \geq 1$.

2. GENERAL PROPERTIES OF $c(n)$

It is instructive to compare the algebra based on centrosymmetry with the orthogonal algebra $so(n)$ based on ordinary symmetry and the pseudo-orthogonal algebra $so(p, n - p)$ (see Gilmore, 1974). Consider the defining relation for an element X of the three algebras:

$$so(n): \quad X^T I + IX = 0 \tag{10a}$$

$$so(p, n - p): \quad X^T L_p + L_p X = 0 \tag{10b}$$

$$c(n): \quad XJ - JX = 0 \tag{10c}$$

where we have used (2a) and (6), and where

$$(L_p)_{ab} = \begin{cases} -\delta_{ab} & (a, b \le p) \\ \delta_{ab} & (a, b > p) \end{cases} \tag{11}$$

The conditions in (10a) and (10b) reflect the invariance of an inner product $\langle \mathbf{x}, \mathbf{y} \rangle_K \equiv \mathbf{x}^T K \mathbf{y}$ on R^n under a Lie algebra g:

$$\langle \cdot, \cdot \rangle_K \quad \text{invariant under} \quad g \Leftrightarrow X^T K + KX = 0 \qquad \text{for all} \quad X \in g$$

For $so(n)$, $K = I$ (Euclidean metric), while for $so(p, n - p)$, $K = L_p$ (pseudo-Euclidean or Lorentzian metric). From the form of (10c), there is no obvious inner product invariant under $c(n)$.

The dimension of $c(n)$ follows from (5) as

$$\dim c(n) = \frac{1}{2}(n^2 + \epsilon), \qquad \epsilon = \begin{cases} 0 & n \text{ even} \\ 1 & n \text{ odd} \end{cases} \tag{12}$$

By (3) we see that $c(n)$ is a *noncompact* algebra. In particular, (4) shows that $c(2)$ contains the Lorentz algebra as a subalgebra:

$$so(1, 1) < c(2) \tag{13}$$

It follows from (8) that

$$(e^X)^C = e^{X^C}$$

so that the local Lie group generated by $c(n)$ is characterized by

$$A \in \exp c(n) \Rightarrow A^C = A \tag{14}$$

Thus $c(n)$ has the unusual property that *it contains the local matrix Lie group which it generates*:

$$\exp c(n) \subset c(n) \tag{15}$$

It follows from (5) and (6) that J commutes with $c(n)$ (so that ad J is null). Thus

$$i_1 \equiv \text{span}\{I\}, \qquad j_1 \equiv \text{span}\{J\}, \qquad k_2 \equiv \text{span}\{I, J\} = i_1 + j_1 \tag{16}$$

are Abelian ideals [furthermore, they are in the center of $c(n)$], and $c(n)$ is consequently *nonsemisimple*. The adjoint representation of $c(n)$ is thus nonfaithful.

By (3), the invariant Abelian subgroup generated by k_2 is

$$\exp k_2 = \{K(\alpha, \beta) \equiv Ie^\alpha \cosh\beta + Je^\alpha \sinh\beta \,|\, \alpha, \beta \in R\} \tag{17}$$

The transformation of R^n by $K(\alpha, \beta)$ follows from (1) and (17) as

$$x'^a = K(\alpha, \beta)^a{}_b x^b = e^\alpha(x^a \cosh\beta + x^{n+1-a} \sinh\beta) \tag{18}$$

For $n = 2m$ ($m \geq 1$), (18) is in the conformal group of the Lorentzian metric L_m, which has signature zero, i.e.,

$$L_m \to L'_m \equiv K(\alpha, \beta) L_m K(\alpha, \beta)^T = e^{2\alpha} L_m \tag{19}$$

Thus: *for n even,* exp k_2 *is a boost-dilation subgroup of the conformal group of the Lorentzian metric with signature* $(- + - + \cdots - +)$.

This conformal property does not extend to $n = 2m + 1$. However, we will show in Section 3 that $c(2m + 1)$ *is nonsolvable.*

3. EXAMPLES

We consider the simplest cases of n even and n odd. For $n = 2$, the algebra reduces to the Abelian ideal (16c), so that, using (4),

$$c(2) = k_2 = i_1 + so(1, 1)$$

which is consistent with (13). By (17), $c(2)$ generates the Abelian group of matrices of form

$$K(\alpha, \beta) = \begin{pmatrix} e^\alpha \cosh \beta & e^\alpha \sinh \beta \\ e^\alpha \sinh \beta & e^\alpha \cosh \beta \end{pmatrix}$$

On the Minkowski plane, $K(\alpha, \beta)$ is a combined Lorentz boost and dilation, as follows from (18):

$$t' = e^\alpha \gamma(v)[t + vx], \qquad x' = e^\alpha \gamma(v)[x + vt] \tag{20}$$

where $\gamma(v) \equiv (1 - v^2)^{-1/2}$ ($c = 1$) is the Lorentz factor and $v = \tanh \beta$ is the boost velocity. By (20), $K(\alpha, \beta)$ transforms the Minkowski metric L_1 in accordance with (19). Thus exp $c(2)$ is a subgroup of the conformal group on the Minkowski plane.

For $n = 3$, using (5) and (6), we find that a natural basis is $c(3) = \text{span}\{X_1, \ldots, X_5\}$, where $X_1 = I$, $X_2 = J$, and

$$X_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{21}$$

The nonzero commutators follow from(21) as

$$[X_3, X_4] = X_1 + X_2 - 4X_5, \qquad [X_3, X_5] = X_3, \qquad [X_4, X_5] = -X_4 \tag{22}$$

Then from (22) it follows that

$$c(3)' = \text{span}\{X_3, X_4, X'_5\}, \qquad X'_5 \equiv X_1 + X_2 - 4X_5 \tag{23}$$

and we can show that

$$c(3)'' \equiv [c(3)', c(3)'] = c(3)' \Rightarrow c(3)^{(m)} = c(3)' \neq 0 \qquad \text{for} \quad m \geq 1$$

Thus the derived series does not terminate, and hence (Gilmore, 1974; Jacobson, 1979) $c(3)$ is nonsolvable.

The two-parameter subgroup (17) has the form

$$K(\alpha, \beta) = \begin{pmatrix} e^{\alpha} \cosh \beta & 0 & e^{\alpha} \sinh \beta \\ 0 & e^{\alpha+\beta} & 0 \\ e^{\alpha} \sinh \beta & 0 & e^{\alpha} \cosh \beta \end{pmatrix} \tag{24}$$

while the one-parameter subgroups generated by (21) are

$$e^{\rho X_3} = \begin{pmatrix} 1 & \rho & 0 \\ 0 & 1 & 0 \\ 0 & \rho & 1 \end{pmatrix}, \qquad e^{\sigma X_4} = \begin{pmatrix} 1 & 0 & 0 \\ \sigma & 1 & \sigma \\ 0 & 0 & 1 \end{pmatrix}, \qquad e^{\mu X_5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\mu} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{25}$$

Then (24) and (25) describe the effect of exp $c(3)$ on R^3. If we consider only $K(\alpha, \beta)$, with $x^a = (t, x, y)$, then, by (18),

$$t' = e^{\alpha}\gamma(v)[t + vy], \qquad x' = e^{\alpha+\beta}x, \qquad y' = e^{\alpha}\gamma(v)[y + vt] \tag{26}$$

Thus $K(\alpha, \beta)$ is a combination of Lorentz boost plus dilation in the t–y plane, with (different) dilation along x. The Minkowski metric is transformed under (26) as

$$L_1 \rightarrow L_1' = e^{2\alpha}L_1 + (e^{2\beta} - 1)X_5$$

so that $K(\alpha, \beta)$ is no longer in the conformal group (except for the trivial case $\beta = 0$).

By (23), we can decompose $c(3)$ as the direct sum

$$c(3) = c(3)' + k_2 \tag{27}$$

We can show that $c(3)'$ is semisimple, in fact isomorphic to $sl(2, R)$, so that, since k_2 is trivially solvable, (27) is the Levi decomposition (Gilmore, 1974) of $c(3)$. Now in the fundamental representation, $sl(2, R) = \text{span}\{Y_1, Y_+, Y_-\}$, where

$$Y_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad Y_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

so that

$$[Y_1, Y_+] = 2Y_+, \qquad [Y_1, Y_-] = -2Y_-, \qquad [Y_+, Y_-] = Y_1 \tag{28}$$

The isomorphism $c(3)' \cong sl(2, R)$ follows on comparing (22) and (28), with the correspondence

$$X_3 \leftrightarrow \frac{1}{\sqrt{2}} Y_+, \qquad X_4 \leftrightarrow -\frac{1}{\sqrt{2}} Y_-, \qquad X_5' \leftrightarrow -\frac{1}{2} Y_1$$

Finally, we can extend the result on nonsolvability from $n = 3$ to $n = 2m + 1$. Consider the basis elements $X_1 = I$, $X_2 = J$, and

$$\begin{aligned} (X_3)_{ab} &= (\delta_{a1} + \delta_{a,2m+1})\delta_{b,m+1} \\ (X_4)_{ab} &= \delta_{a,m+1}(\delta_{b1} + \delta_{b,2m+1}) \\ (X_5)_{ab} &= \delta_{a,m+1}\delta_{b,m+1} \end{aligned} \tag{29}$$

which are of the same form as (21). Using (29), we find

$$[X_3, X_5] = X_3, \qquad [X_4, X_5] = -X_4, \qquad [X_3, [X_3, X_4]] = -4X_3 \tag{30}$$

Then (30) implies that $X_3 \in c(2m + 1)'$ and that its adjoint operator is not nilpotent, i.e., $(\mathrm{ad}\, X_3)^2 \neq 0$. By Engels' theorem (Jacobson, 1979), this proves that $c(2m + 1)$ is nonsolvable.

In conclusion, the centrosymmetric Lie algebra $c(n)$ has some interesting mathematical properties. Although there appears to be no inner product invariant under $c(n)$, and therefore no obvious application of $c(n)$ as a classical or quantum symmetry algebra, it does generate boost/dilations, and $c(2m)$ generates a subgroup of the conformal group of the Lorentzian metric with signature zero. We note also that centrosymmetric matrices arise as transition matrices for certain Markov processes, and that symmetric Toeplitz matrices are also centrosymmetric, and emerge in approximations to the kernels of certain integral equations (Weaver, 1985).

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